Taylor Tricks

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- practicing for midterm: textbook problems, pset problems, practice midterm

taylor's thm: $f: \mathbb{R} \to \mathbb{R}$ differentiable @ $x = a$ then...

$$* \underset{x \approx a}{f(x) \uparrow =} f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots = \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(a)}{n!}}_{a_n} (x-a)^n *$$

simple examples: suppose $f(x)$ = polynomial ($f = 1 - x^2 + 3x^5$)

1) taylor series @ $x = 0$ is itself (it's finite!) * it @ $x = a$, still finite (but rewritten) *
put $(x-a)$ in

2) taylor series, when convergent, sum ($\pm$) & multiply ($\cdot$ $\div$)

ex) taylor of $x^3 e^x \underset{x=0}{=} x^3\left(1 + x + \frac{x^2}{2!} + \ldots\right) = x^3 + x^4 + \frac{x^5}{2!} + \frac{x^6}{3!} = \sum_{n=0}^{\infty} \frac{x^{n+3}}{n!}$

taylor for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$

taylor for $x^3 = x^3$

ex 1) $f(x) = \cos(x)$ @ $x = 0$, its taylor expansion is ...

$$\cos(x) \overset{x \approx 0}{=} 1 - \underbrace{\frac{\sin 0}{1!}}_{0} x + \frac{-1}{2!} x^2 + \underbrace{\frac{\sin 0}{3!}}_{0} x^3 + \frac{1}{4!} x^4 + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \ldots$$ (memorize)

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}$$ ← this is the series

$f(0) = \cos(0) = 1$

$f'(0) = -\sin(0) = 0$

$f''(0) = -\cos(0) = -1$

$f'''(0) = \sin(0) = 0$

$f^{(4)}(0) = \cos(0) = 1$

⋮ repeats

* even function: reflects over y-axis / only even powers

odd function: reflects over origin / only odd powers *

useful tricks:

1) taylor series, when convergent, differentiate & integrate well → easy on RHS (power series)

$\left(f(x) = \sum \text{taylor series} \rightsquigarrow f'(x) = (\sum \text{taylor})' = \sum \text{taylor}'\right)$

ex) $\underbrace{\sin(x) \overset{x=0}{=} [-\cos(x)]'}_{\text{derivative}} = \left(-1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots\right)'$

integral

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}$

2) taylor series compose well (when convergent)

ex) $\sin(x^4) \overset{x=0}{=} x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!} - \frac{x^{28}}{7!} + \ldots$

↳ $\sin(\square) = \square - \frac{\square^3}{3!} + \frac{\square^5}{5!} - \frac{\square^7}{7!} + \ldots$ $\quad \square = x^4$

remark: $\sin(e^x)$? still plug, but now need to expand

↳ $\square = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)^3$ & so on remark: $\sin(e^x)$? still plug, but now need to expand

↳ $\square = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)^3$ & so on

learn later